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A Trial of the Gauge Theory for the Stress-Function Space

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Following Golebiewska-Lasota and Edelen, a gauge structure existing in the field of the stress and the stress function is demonstrated, on the basis of an electromagnetic analogy given elsewhere. The four-tensors and four-vectors are used to state the set of fundamental equations, and the gauge condition and the gauge transformations are given in a similar way to electromagnetic theory.

1. INTRODUCTION

Golebiewska-Lasota (1979) demonstrated the gauge transformation structure underlying in the dynamics of dislocations in solids. The theory was extended by Golebiewska-Lasota and Edelen (1979) to the problem of defects including disclinations, and other authors (Kunin and Kunin, 1986; Kröner, 1986; Edelen and Lagoudas, 1987; Kadič and Edelen, 1983) to the general gauge theory of defects of the Yang–Mills type.

By starting with the fact that the relation between stress and stress functions is similar to the one between incompatibility and strains, the geometrical theory of stress functions was developed and extended to the theory of non-Riemannian stress-function space, where the stress and couple stress are referred to the curvature and torsion tensors, respectively (Schaefer, 1953; Minagawa, 1962, 1968). This corresponds to the geometrical theory of defects by non-Riemannian geometry, where we use the strain space having the incompatibility and dislocation density as those geometrical terms.

It has been shown that we have a principle of duality between those spaces so that the expression including those geometrical terms is converted into another one, having a certain physical meaning, by the substitution of those terms for their corresponding ones in the original. For example, the

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formula for the force exerted on dislocations by external stresses has the structure

(the curvature tensor of the stress-function space)
$\times$ (the torsion tensor of the strain space)

This is converted into

(the curvature tensor of the strain space)
$\times$ (the torsion tensor of the stress-function space)

which has the meaning of the force exerted on the incompatibility by the external couple stresses (Minagawa, 1970).

As has been mentioned, the gauge transformation structure underlies the defect dynamics. The gauge theory of Golebiewska-Lasota and Edelen is in the framework of the theory of strain space; it is converted into the framework of the theory of stress-function space. The aim of this paper is to advance a first step in this direction.

2. BASIC EQUATIONS

We assume an orthogonal Cartesian coordinate system with respect to which the position of a material point is x_i. Throughout this paper, lower case Roman indices $i, j, k, \ldots$ take values 1, 2, or 3, and Einstein's summation convention is used for those indices appearing twice in one expression. A superposed dot means a derivative with regard to the time t.

We have the following set of basic equations (Minagawa, 1971):

(a) Equations of continuity of dislocations:

$$\partial_j \alpha_{ji} = R_i \tag{1}$$

(b) Equations of balance of dislocations:

$$\varepsilon_{jmn}\, \partial_m \dot{\beta}^P_{ni} = \frac{\partial \alpha_{ji}}{\partial t} + R_i u_j \tag{2}$$

(c) Equations relating the stress and stress functions:

$$\varepsilon_{jmn}\, \partial_m \Sigma_{in} = -\frac{\partial \Psi_{ij}}{\partial t} + \sigma_{ij} \tag{3}$$

(d) Subsidiary conditions for the stress functions:

$$\partial_j \Psi_{ij} = \rho w_i \tag{4}$$

where ε_{ijk} is Eddington's epsilon, α_{ij} the dislocation-density tensor, R_i the density of dislocation sources, β^P_{ij} the plastic distortion tensor, Σ_{ij} and Ψ_{ij} the stress functions, σ_{ij} the stress tensor, u_i the velocity of dislocation source, w_i the velocity of displacement, and ρ the mass density.

As given in Minagawa (1971), equations (1)-(4) have the same form as the Maxwell equations in electromagnetic theory, and therefore the terms of dislocations and stresses are compared with the terms of electricity and magnetism. Based on this analogy, we shall state the basic equations of dislocations and stresses in terms of the four-tensors and four-vectors to lead to the gauge conditions and gauge transformations which correspond to those in the electromagnetic theory.

3. FOUR-TENSOR REPRESENTATIONS

We introduce the system of space-time coordinates (x, ict), where i equals $\sqrt{-1}$ and c is a parameter having the dimension of the velocity. The upper case Roman indices $I, J, K, \ldots$ take values 1, 2, 3, 4, and Einstein's summation convention is used also for those indices. The index 4 corresponds to the time coordinate.

By assuming the four-tensor and four-vector such that

$$H^p_{IJ}: \begin{bmatrix} 0 & \alpha_{3p} & -\alpha_{2p} & \frac{i}{c}\dot{\beta}^P_{1p} \\ -\alpha_{3p} & 0 & \alpha_{1p} & \frac{i}{c}\dot{\beta}^P_{2p} \\ \alpha_{2p} & -\alpha_{1p} & 0 & \frac{i}{c}\dot{\beta}^P_{3p} \\ -\frac{i}{c}\dot{\beta}^P_{1p} & -\frac{i}{c}\dot{\beta}^P_{2p} & -\frac{i}{c}\dot{\beta}^P_{3p} & 0 \end{bmatrix} \tag{5}$$

and

$$N^p_J: \begin{bmatrix} \frac{i}{c}R_p u_1 & \frac{i}{c}R_p u_2 & \frac{i}{c}R_p u_3 & R_p \end{bmatrix} \tag{6}$$

We can write equations (1) and (2)

$$\frac{\partial H^p_{IJ}}{\partial x_K}+\frac{\partial H^p_{JK}}{\partial x_I}+\frac{\partial H^p_{KI}}{\partial x_J}=N^p_L \tag{7}$$

where (I, J, K, L) is one of the even permutations of $(1, 2, 3, 4)$.

On the other hand, if we put

$$G^p_{IJ}: \begin{bmatrix} 0 & \Sigma_{p3} & -\Sigma_{p2} & ic\Psi_{p1} \\ -\Sigma_{p3} & 0 & \Sigma_{p1} & ic\Psi_{p2} \\ \Sigma_{p2} & -\Sigma_{p1} & 0 & ic\Psi_{p3} \\ -ic\Psi_{p1} & -ic\Psi_{p2} & -ic\Psi_{p3} & 0 \end{bmatrix} \tag{8}$$

and

$$M_I^p: \quad [\sigma_{p1} \quad \sigma_{p2} \quad \sigma_{p3} \quad ic\rho w_p] \tag{9}$$

we can combine equations (3) and (4) into

$$\frac{\partial G_{IJ}^p}{\partial x_J} = M_I^p \tag{10}$$

for $I = 1, 2, 3$, or 4. If $J = 4$, we have equation (3), and if $J = 1$, 2, or 3, equation (4) is given.

4. GAUGE CONDITIONS

Since G_{IJ}^p is antisymmetric with respect to J and K,

$$\frac{\partial M_J^p}{\partial x_J} = \frac{\partial^2 G_{JK}^p}{\partial x_J \, \partial x_K} = 0 \tag{11}$$

is identically given. The last equation leads to

$$\partial_j \sigma_{pj} = \rho \dot{w}_p \tag{12}$$

which is the equation of motion of the materials.

Corresponding to the Lorentz gauge condition in the theory of electrodynamics, we introduce the condition

$$\frac{\partial G_{JK}^p}{\partial x_I} + \frac{\partial G_{KI}^p}{\partial x_J} + \frac{\partial G_{IJ}^p}{\partial x_K} = 0 \tag{13}$$

By taking $(1, 2, 3)$ as (I, J, K), we can reduce the last equation to

$$\frac{\partial \Sigma_{pn}}{\partial x_n} = 0 \tag{14}$$

and $(4, 2, 3)$, etc.,

$$\varepsilon_{jmn} \, \partial_m \Psi_{pn} - \frac{1}{c^2} \frac{\partial \Sigma_{pj}}{\partial t} = 0 \tag{15}$$

The last two equations imply the subsidiary conditions which are imposed on the stress functions. We have used those subsidiary conditions in the computation of the stress field produced by moving dislocations.

5. GAUGE TRANSFORMATIONS

If we substitute

$$\bar{\Psi}_{pj} = \Psi_{pj} + B_{pj} \tag{16}$$

equation (4) is reduced to

$$\rho w_p = \partial_j \bar{\Psi}_{pj} - \partial_j B_{pj} \tag{17}$$

Thus, the new variable $\bar{\Psi}_{pj}$ satisfies equation (4) if and only if $\partial_j B_{pj}$ vanishes; i.e., we have

$$B_{pj} = -\varepsilon_{jmn}\,\partial_m b_{pn} \tag{18}$$

where b_{pn} is a tensor function of the space and time coordinates.

Next, we substitute

$$\bar{\Sigma}_{pn} = \Sigma_{pn} + A_{pn} \tag{19}$$

in equation (3) to obtain

$$\sigma_{pj} = \varepsilon_{jmn}\,\partial_m \bar{\Sigma}_{pn} + \dot{\bar{\Psi}}_{pj} - \varepsilon_{jmn}\,\partial_m (A_{pn} - \dot{b}_{pn}) \tag{20}$$

The invariance of the stress field is guaranteed by the vanishing of the last term of the right-hand side of equation (20), so that we have

$$A_{pn} - \dot{b}_{pn} = \partial_n a_p \tag{21}$$

where a_p is a vector function of the space and time coordinates.

Thus we arrive at the gauge transformations

$$\begin{aligned} \Sigma_{pn} \to \bar{\Sigma}_{pn} &= \Sigma_{pn} + \partial_n a_p + \dot{b}_{pn} \\ \Psi_{pj} \to \bar{\Psi}_{pj} &= \Psi_{pj} - \varepsilon_{jmn}\,\partial_m b_{pn} \end{aligned} \tag{22}$$

which map equations (3) and (4) into themselves and leave the M_{IJ}^p field invariant.

In space-time terminology, the transformation (22) is stated as follows:

$$G_{IJ}^p \to \bar{G}_{IJ}^p = G_{IJ}^p + \Gamma_{IJ}^p, \qquad \Gamma_{IJ}^p = ic\varepsilon_{IJKL} \frac{\partial \Phi_K^p}{\partial x_L} \tag{23}$$

where ε_{IJKL} is Levi-Civita symbol and Φ_I^p are given by

$$\Phi_i^p = b_{pi}, \qquad \Phi_4^p = \frac{i}{c}\, a_p \tag{24}$$

which corresponds to the four-potential in the electromagnetic theory.

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